M·Sc IVth SEM
2020

Curvature —

The Curvature at a point P of a given curve is the arc rate of rotation of tangent at P. Its magnitude is denoted by K.

Torsion —

Torsion at point P of a given curve is the arc rate of the change in the direction of the binomial at P. Its magnitude is denoted by τ.

To find Curvature and Torsion of a curve -

Case I — when $r = f(u)$

$$\dot{r} = r'\dot{s}, \quad \dot{t} = t'\dot{s}$$
$$\dot{t} = \dot{s}Kn$$
$$\dot{n} = n'\dot{s} = \dot{s}(\tau b - Kt)$$
$$\dot{b} = b'\dot{s} = -\dot{s}\tau n$$

Again $\dot{r} = \dot{s}t$, $\ddot{r} = \ddot{s}t + \dot{s}^2 Kn$

$$\dddot{r} + \dddot{s}t = \ddot{s}\dot{s}Kn + 2\dot{s}\ddot{s}Kn + \dot{s}^2\dot{K}n + \dot{s}^3 K(\tau b - Kt)$$

Date
Page

Obviously

$$\dot{r} \times \ddot{r} = \dot{s}^3 k b \quad \text{——} ①$$

$$[\dot{r}, \ddot{r}, \dddot{r}] = (\dot{r} \times \ddot{r}) \cdot \dddot{r} = \dot{s}^6 k^2 \tau \quad \text{——} ②$$

Also $$|\dot{r}| = |\dot{s}t| = \sqrt{\dot{s}^2} = \dot{s} \quad \text{——} ③$$

$$|\dot{r} \times \ddot{r}| = \dot{s}^3 k \quad \text{——} ④$$

$\therefore$ from eq (3) and (4)

$$k = \frac{|\dot{r} \times \ddot{r}|}{\dot{s}^3} = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3} \quad \text{——} ⑤$$

from eq (1) and (2)

$$\tau = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{[\dot{r} \times \ddot{r}]^2}$$

Examples

1). Calculate the Curvature and torsion of the cubic curve given by $r = (u, u^2, u^3)$.

Solution — Here $r = (u, u^2, u^3)$

$\therefore$ $r = (1, 2u, 3u^2)$

$\ddot{r} = (0, 2, 6u)$

Scanned by CamScanner

$$\dddot{r} = (0, 0, 6)$$

$$\dot{r} \times \ddot{r} = (1 + 2uj + 3u^2k) \times (2j + 6uk)$$

$$= 2k - 6uj + 12u^2 i - 6u^2 i$$

$$= 6u^2 i - 6uj + 2k$$

$$= (6u^2, -6u, 2)$$

$$= 2(3u^2, -3u, 1)$$

$\therefore$ $$[\dot{r} \times \ddot{r}] = 2(9u^4 + 9u^2 + 1)^{1/2}$$

~~Also~~ $$[\dot{r}, \ddot{r}, \dddot{r}] = \dot{r} \times \ddot{r} \cdot \dddot{r}$$
$$= 2(3u^2, -3u, 1) \cdot (0, 0, 6)$$
$$= 2(0 + 0 + 6)$$
$$= 12$$

$$\kappa = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3} = \frac{2(9u^2 + 9u^2 + 1)^{1/2}}{(1 + 4u^2 + 9u^4)^{3/2}}$$

$$\tau = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{[\dot{r} \times \ddot{r}]^2} = \frac{12}{4(9u^4 + 9u^2 + 1)}$$

$$\boxed{\tau = \frac{3}{(9u^4 + 9u^2 + 1)}}$$

Example To find the radii of curvature and torsion of the helix

$$x = a\cos u$$
$$y = a\sin u$$
$$z = a\,u\tan\alpha$$

Solution — Here $r = (a\cos u, a\sin u, au\tan\alpha)$

Differentiating w.r.t 'u'

$$\dot{r} = a(-\sin u, \cos u, \tan\alpha)$$
$$\ddot{r} = a(-\cos u, -\sin u, 0)$$
$$\dddot{r} = a(\sin u, -\cos u, 0)$$

$$\dot{r} \times \ddot{r} = a^2(\sin u\tan\alpha, -\cos u\tan\alpha, 1)$$

$$|\dot{r} \times \ddot{r}| = a^2[\sin^2 u\tan^2\alpha + \cos^2 u\tan^2\alpha + 1]^{1/2}$$

$$= a^2\sec\alpha$$

$$|\dot{r}, \ddot{r}, \dddot{r}| = \dot{r} \times \ddot{r} \cdot \dddot{r}$$
$$= a^3(\sin^2 u\tan\alpha + \cos^2 u\tan\alpha + 0)$$
$$= a^3\tan\alpha$$

$\therefore$ Also $|\dot{r}| = a[\sin^2 u + \cos^2 u + \tan^2\alpha]^{1/2}$
$$= a\sec\alpha$$

$\therefore$

$$\kappa = \frac{a^2\sec\alpha}{a^3\sec^3\alpha}$$

$$= \frac{1}{a\sec^2\alpha} = \frac{\cos^2\alpha}{a}$$

$$\therefore \quad \rho = \frac{1}{\kappa} = a\sec^2\alpha$$

$$\tau = \frac{a^3\tan\alpha}{a^4\sec^2\alpha}$$

$$= \frac{\sin\alpha}{a\sec\alpha}$$

$$\therefore \quad \sigma = \frac{1}{\tau} = a\,\mathrm{cosec}\,\alpha\,\sec\alpha$$

Scanned by CamScanner